

# THE EXTENDING FOR COMPOSITE SKYRME MODEL

Pham Thuc Tuyen <sup>1</sup>, Do Quoc Tuan <sup>2</sup>

*Department of Physics, Hanoi University of Science,  
334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam.*

**Abstract.** *In this paper, we have extended the composite Skyrme model proposed by H. Y. Cheung, F. Gursey to some respects of theoretical physics such as supersymmetry, gravitation.*

*Keywords: Skyrme model, QCD, supersymmetry, gravitation, Einstein equation.*

## 1 Introduction

The Skyrme model [2] was proposed to repair weak points of QCD at low energy (in GeV). Time by time, it really become a phenomenological model of baryons. In 1990, H. Y. Cheung, F. Gursey [1] gave the general Skyrme model in which the sigma model [3] and the  $V - T$  model [9] were two special cases corresponding to  $n=1$  and  $n=2$ . When Cheung and Gursey constructed composite Skyrme model, they also didn't let the mass of pion into the Lagrangian density [1] because of its spontaneous symmetry breaking [3, 4]. However, datum of prediction are not really axact. However, datum of prediction are not really axact [1]. So, we have proposed the Lagrangian density with the pion's mass. In 1984, E. A. Bergshoeff, R. I. Nepomachie, H. J. Schnitzer proposed Skyrmon of four-dimensional supersymmetric non-linear sigma model [5]. However, there were not papers that concerned with an extending the composite Skyrme model to the supersymmetry. Thus, we have performed to extend the commposite Skyrme model into supersymmetry. Recently, the Einstein-Skyrme system has been studied by several authors [6, 7, 8]. And then the skyrmion to be the gravitating skyrmion. All of these study were related to the non-linear Skyrme model but were not related to the composite Skyrme model. Thus, we have constructed the gravitating composite skyrmion.

## 2 The mass of pion

The Lagrangian density is defined as

$$\mathcal{L}_n^{pion} = \mathcal{L}_n + \frac{1}{8n} m_\pi^2 F_\pi^2 [Tr(U^n) - 2]. \quad (1)$$

Putting the hedgehog solution given by Skyrme [2]  $U_o(r) = \exp[i\tau \cdot \hat{r} F(r)]$  into the Lagrangian density (1), we have

$$\mathcal{L}_n^{pion} = \mathcal{L}_n + \frac{1}{4n} m_\pi^2 F_\pi^2 (\cos nF - 1), \quad (2)$$

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<sup>1</sup>tuyenpt@coltech.vnu.vn (or) phamthuc.tuyen@gmail.com

<sup>2</sup>do.tocxoan@gmail.com

where  $\tau$ 's are Pauli's matrices,  $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$  and

$$\mathcal{L}_n = \frac{F_\pi^2}{8} \left[ \left( \frac{dF}{dr} \right)^2 + 2 \frac{\sin^2 nF}{n^2 r^2} \right] + \frac{\sin^2 nF}{2e^2 n^2 r^2} \left[ 2 \left( \frac{dF}{dr} \right)^2 + \frac{\sin^2 nF}{n^2 r^2} \right]. \quad (3)$$

The static energy of chiral field is

$$\mathcal{E}_n^{pion} = \frac{F_\pi}{e} A_n + \frac{1}{e^3 F_\pi} \Theta_n, \quad (4)$$

where

$$A_n = \int_0^\infty 4\pi \tilde{r}^2 \left\{ \left[ \frac{F'^2}{8} + \frac{\sin^2 nF}{4n^2 \tilde{r}^2} \right] + \frac{e^2 F_\pi^4 \sin^2 nF}{n^2 \tilde{r}^2} \left[ F'^2 + \frac{\sin^2 nF}{2n^2 \tilde{r}^2} \right] \right\} d\tilde{r}, \quad (5)$$

$$\Theta_n = \int_0^\infty \frac{\pi \tilde{r}^2 m_\pi^2}{n} (\cos nF - 1) d\tilde{r}, \quad (6)$$

and  $\tilde{r} = eF_\pi r$  is a dimensionless variable.

Now, we shall quantize the hedgehog solution ( or *soliton*) by collective coordinates  $A(t) = a_o(t) + i\vec{a}(t) \cdot \vec{\tau}$  with coditions  $a_\mu a_\mu = 1$  and  $a_\mu \dot{a}_\mu = 0$  ( $\mu = 0, 1, 2, 3$ ). By this way, solitons being real partilces. Different quantizations make different real particles. With the  $A(t)$  matrix, the  $U^n(x, t)$  will transform as  $U^n(x, t) = A(t) U_0^n(x) A^{-1}(t)$ . We put  $U^n(x, t)$  into Eq. (1) then the Lagrangian is

$$L_n = -\mathcal{E}_n^{pion} + \Gamma_n Tr [\partial_0 A(t) \partial_0 A^{-1}(t)] = -\mathcal{E}_n^{pion} + 2\Gamma_n \sum_{\mu=0}^3 \dot{a}_\mu^2, \quad (7)$$

where

$$\Gamma_n = \frac{1}{e^3 F_\pi} \Phi_n, \quad (8)$$

$$\Phi_n = \frac{2\pi}{3n^2} \int_0^\infty \tilde{r}^2 \sin^2 nF \left\{ 1 + 4 \left[ \left( \frac{dF}{d\tilde{r}} \right)^2 + \frac{\sin^2 nF}{n^2 \tilde{r}^2} \right] \right\} d\tilde{r}. \quad (9)$$

The conjugate momenta are defined as

$$\pi_\mu = \frac{\partial L_n}{\partial \dot{a}_\mu} = \partial \left[ -\mathcal{E}_n^{pion} + 2\Gamma_n \sum_{\mu=0}^3 \dot{a}_\mu^2 \right] / \partial \dot{a}_\mu = 4\Gamma_n \dot{a}_\mu. \quad (10)$$

From Eq. (7), we can get the Hamiltonian

$$H_n = \pi_\mu \dot{a}_\mu - L_n = 4\Gamma_n \dot{a}_\mu \dot{a}_\mu - L_n = \mathcal{E}_n^{pion} + 2\Gamma_n \dot{a}_\mu \dot{a}_\mu = \mathcal{E}_n^{pion} + \left( \sum_{\mu=0}^3 \pi_\mu^2 \right) / 8\Gamma_n. \quad (11)$$

Canonical quantizing momenta  $\pi_\mu = -i\partial/\partial a_\mu$  takes the Hamiltonian as

$$H_n = \mathcal{E}_n^{pion} + (1/8\Gamma_n) \sum_{\mu=0}^3 (-\partial^2/\partial a_\mu^2). \quad (12)$$

The isotopic vector  $\vec{c}$  is defined as

$$\dot{A}^\dagger A = (\dot{a}_0 - i\dot{a}_i\tau_i)(a_0 + ia_i\tau_i) = \dot{a}_0a_0 + \dot{\vec{a}}\vec{a} - i\left(a_0\dot{\vec{a}} - \dot{a}_0\vec{a} + \vec{a} \times \dot{\vec{a}}\right)\vec{\tau} = -i\vec{c}\vec{\tau}. \quad (13)$$

The square isotopic vector  $\vec{c}$  is

$$\vec{c}^2 = \vec{a}^2\dot{a}_0^2 + \dot{\vec{a}}^2a_0^2 + \vec{a}^2\dot{\vec{a}}^2 - (\dot{\vec{a}}\vec{a})^2 - 2\dot{a}_0a_0\dot{\vec{a}}\vec{a} = \vec{a}^2\dot{a}_0^2 + \dot{\vec{a}}^2a_0^2 + \vec{a}^2\dot{\vec{a}}^2 - (\dot{\vec{a}}\vec{a})^2 + 2(\dot{\vec{a}}\vec{a})^2 = \dot{a}_\mu\dot{a}_\mu. \quad (14)$$

In the configuration of hedgehog, the spin and isospin rotation are equivalent. We can homogenize  $\vec{c}^2$  with the square of spin and isospin

$$\vec{c}^2 = \frac{1}{16\Gamma_n^2}\vec{J}^2 = \frac{1}{16\Gamma_n^2}\vec{T}^2. \quad (15)$$

Thus, we have the Hamiltonian

$$H_n = \mathcal{E}_n^{pion} + \frac{1}{8\Gamma_n}\vec{J}^2 = \mathcal{E}_n^{pion} + \frac{1}{8\Gamma_n}\vec{T}^2, \quad (16)$$

where  $\vec{J}$  is a spin and  $\vec{T}$  is an isospin. Eigenvalues of Hamiltonian (16) are

$$M_n = E_n = \frac{F_\pi}{e}A_n + \frac{1}{e^3F_\pi}\Theta_n + \frac{l(l+2)}{8\Gamma_n}, \quad (17)$$

where  $l = 2J = 2I$  and  $(J, I)$  are respectively the isospin and spin quantum numbers. So, the mass of nucleon ( $J = 1/2$ ) and delta ( $J = 3/2$ ) are given by

$$M_N = \frac{F_\pi}{e}A_n + \frac{1}{e^3F_\pi}\Theta_n + \frac{3e^3F_\pi}{8\Phi_n}, \quad (18)$$

$$M_\Delta = \frac{F_\pi}{e}A_n + \frac{1}{e^3F_\pi}\Theta_n + \frac{15e^3F_\pi}{8\Phi_n}. \quad (19)$$

From Eqs. (18) and (19), we can get

$$F_\pi = \frac{2(M_\Delta - M_N)\Phi_n}{3e^3}, \quad (20)$$

$$e = \sqrt[4]{\frac{4(M_\Delta - M_N)^3\Phi_n^2A_n}{6M_N(M_\Delta - M_N)^2\Phi_n - 9\Theta_n - 3(M_\Delta - M_N)^2\Phi_n}}. \quad (21)$$

We obtain the nonlinear differential equation of  $F(\tilde{r})$  from the minimum condition of the chiral field's energy:  $\delta_F \mathcal{E}_n^{pion} = 0$

$$\left(\frac{\tilde{r}^2}{4} + \frac{2}{n^2}\sin^2 nF\right)F'' + \frac{\tilde{r}}{2}F' + \frac{\sin 2nF}{n}\left(F'^2 - \frac{\sin^2 nF}{n^2\tilde{r}^2} - \frac{1}{4}\right) - \beta\tilde{r}^2\sin nF = 0, \quad (22)$$

where

$$\beta = \frac{m_\pi^2}{4e^2F_\pi^2}. \quad (23)$$

We see that Eq. (23) contain the constant of  $e$  and  $F_\pi$  but their values are not known. They can be found after we solve Eq. (22). Thus, we must use the special way to solve Eq. (22). Firstly, we assume the value of  $\beta^*$ , put it into Eq. (22) to solve this equation. When numerical solution of Eq. (22) is found, values of  $F_\pi$ ,  $e$  and  $\beta^{**}$  will be found from Eqs. (20), (21) and (23). After, we shall compare two  $\beta^*$  and  $\beta^{**}$  until they equal approximately together. For the case of  $n = 4$ , we need the value of  $\beta$  is 0.0025.

After constructing coposite Skyrme model with pion's mass, we shall apply this model to calculate some static properties of nucleon (proton and neutron) [1] in a case of  $n = 4$ . And numerical results of static properties of nucleon are described in Fig. (2) and the below table

NUMERICAL RESULTS TABLE IN A CASE OF N=4

Quantity	Prediction	Experiment
$e$	2.87	-
$F_\pi(\text{MeV})$	446.56	186
$\langle r^2 \rangle_{E,I=0}^{1/2}(\text{fm})$	0.21	0.71
$\langle r^2 \rangle_{E,I=1}^{1/2}(\text{fm})$	2.26	0.88
$\langle r^2 \rangle_{M,I=0}^{1/2}(\text{fm})$	1.0	0.81
$\langle r^2 \rangle_{M,I=1}^{1/2}(\text{fm})$	2.26	0.8
$\mu_{pro}(\text{mag})$	1.64	2.79
$\mu_{neu}(\text{mag})$	-1.57	-1.91
$\left  \frac{\mu_{pro}}{\mu_{neu}} \right $	1.04	1.46
$g_A$	1.22	1.23

### 3 The supersymmetric composite Skyrme model

The Lagrangian density of composite Skyrme model is defined as

$$\mathcal{L}_n = -\frac{f_\pi^2}{16n^2} \text{Tr} (\partial_\mu U^{-n} \partial^\mu U^n) + \frac{1}{32e^2 n^4} \text{Tr} ([U^{-n} \partial_\mu U^n, U^{-n} \partial_\nu U^n]^2), \quad (24)$$

where  $f_\pi$  is the pi-meson decay constant,  $e$  is a dimensionless parameter.

The ordinary derivatives in the Lagrangian density (24) is replaced into the covariant derivatives

$$D_\mu U^n = \partial_\mu U^n - iV_\mu U^n \tau_3, \quad (25)$$

then Eq. (24) is

$$\mathcal{L}_n = -\frac{f_\pi^2}{16n^2} \text{Tr} (D_\mu U^{-n} D^\mu U^n) + \frac{1}{32e^2 n^4} \text{Tr} ([U^{-n} D_\mu U^n, U^{-n} D_\nu U^n]^2). \quad (26)$$

Eq. (26) is invariant under the local  $U(1)_R$  and the global  $SU(2)_L$  transformations

$$U^n(r) \rightarrow AU^n(x) e^{i\lambda(r)\tau_3}, A \in SU(2)_L, \quad (27)$$

$$V_\mu(r) \rightarrow V_\mu(r) + \partial_\mu \lambda(r). \quad (28)$$

The gauge field  $V_\mu(r)$  in Eq. (28) is defined as

$$V_\mu = -\frac{i}{2n} Tr(U^{-n} \partial_\mu U^n \tau_3). \quad (29)$$

We parametrize the  $SU(2)$  matrix  $U^n(r)$  in terms of the complex scalars  $A_i$

$$U^n(r) = \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}. \quad (30)$$

From the unitary constraint, we have

$$U^\dagger U = 1 \Rightarrow \bar{A}^i A_i = A_1^* A_1 + A_2^* A_2 = 1. \quad (31)$$

Eq. (25) can be rewritten as

$$D_\mu \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix} = \partial_\mu \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix} - iV_\mu \begin{pmatrix} A_1 & A_2^* \\ A_2 & -A_1^* \end{pmatrix} \quad (32)$$

or

$$D_\mu A_i = (\partial_\mu - iV_\mu) A_i, \quad (33)$$

$$D_\mu \bar{A}_i = (\partial_\mu + iV_\mu) \bar{A}_i. \quad (34)$$

Then, the form of gauge field (29) is

$$V_\mu(r) = -\frac{i}{2n} \bar{A}^i \vec{\partial} A_i. \quad (35)$$

With the matrix  $U^n(r)$  defined in Eq. (30) and the gauge field defined in Eq. (35), the supersymmetric Lagrangian density is

$$\mathcal{L}_n = -\frac{f_\pi^2}{8n^2} \bar{D}_\mu \bar{A} D^\mu A - \frac{1}{16e^2 n^2} F_{\mu\nu}^2(V), \quad (36)$$

where

$$F_{\mu\nu}(V) = \partial_\mu V_\nu - \partial_\nu V_\mu. \quad (37)$$

To supersymmetrise composite Skyrme model, we extend  $A_i$  to chiral scalar multiplets  $(A_i, \psi_{\alpha i}, F_i)$  ( $i, \alpha = 1, 2$ ) and the vector  $V_\mu(r)$  to real vector multiplets  $(V_\mu, \lambda_\alpha, D)$ . Here, the fields  $F_i$  are complex scalars,  $D$  is real scalar,  $\psi_{\alpha i}$ ,  $\lambda_\alpha$  are Majorana two-component spinors.  $\psi_{\alpha i}$  corresponds to a left-handed chiral spinor,  $\bar{\psi}^{\alpha i} = (\psi_i^\alpha)^*$  corresponds to a right-handed one. The supersymmetric Lagrangian density is given by

$$\begin{aligned}
\mathcal{L}_{susy} = & \frac{f_\pi^2}{8n^2} \left[ -D^\mu \bar{A}^i D_\mu A_i - \frac{1}{2} i \bar{\psi}^{\dot{\alpha}i} (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{D}^\mu \psi_i^\alpha + \bar{F}^i F_i - \right. \\
& \left. - i \bar{A}^i \lambda^\alpha \psi_{\alpha i} + i A_i \bar{\lambda}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}^i + D (\bar{A}^i A_i - 1) \right] + \\
& + \frac{1}{8e^2 n^2} \left[ -\frac{1}{2} F_{\mu\nu}^2 - i \bar{\lambda}^{\dot{\alpha}} (\sigma^\mu)_{\dot{\alpha}}^\alpha \partial_\mu \lambda_\alpha + D^2 \right].
\end{aligned} \tag{38}$$

It is invariant under the following set of supersymmetric transformations [5]

$$\delta A_i = -\varepsilon^\alpha \psi_{\alpha i}, \tag{39}$$

$$\delta \psi_{\alpha i} = -i \bar{\varepsilon}^{\dot{\alpha}} (\sigma^\mu)_{\alpha\dot{\alpha}} D_\mu A_i + \varepsilon_\alpha F_i, \tag{40}$$

$$\delta F_i = -i \bar{\varepsilon}^{\dot{\alpha}} (\sigma^\mu)_{\dot{\alpha}}^\alpha D_\mu \psi_{\alpha i} - i \bar{\varepsilon}^{\dot{\alpha}} A_i \bar{\lambda}_{\dot{\alpha}}, \tag{41}$$

$$\delta V_\mu = -\frac{1}{2} i (\sigma_\mu)^{\alpha\dot{\alpha}} (\bar{\varepsilon}_{\dot{\alpha}} \lambda_\alpha + \varepsilon_\alpha \bar{\lambda}_{\dot{\alpha}}), \tag{42}$$

$$\delta \lambda_\alpha = \varepsilon^\beta (\sigma^{\mu\nu})_{\beta\alpha} F_{\mu\nu} + i \varepsilon_\alpha D, \tag{43}$$

$$\delta D = \frac{1}{2} (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu (\bar{\varepsilon}^{\dot{\alpha}} \lambda^\alpha - \varepsilon^\alpha \bar{\lambda}^{\dot{\alpha}}). \tag{44}$$

The field equations and their supersymmetric transformations lead to the following constraints [5]

$$\bar{A}^i A_i = 0, \tag{45}$$

$$\bar{A}^i \psi_{\alpha i} = 0, \tag{46}$$

$$\bar{A}^i F_i = 0 \tag{47}$$

and following algebraic expressions [5]

$$V_\mu = -\frac{1}{2} \left\{ i \bar{A}^i \bar{\partial}_\mu A_i + (\sigma_\mu)^{\alpha\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}^i \psi_{\alpha i} \right\}, \tag{48}$$

$$\lambda_\alpha = -i \left\{ \bar{F}^i \psi_{\alpha i} + i (\sigma^\mu)_{\alpha\dot{\alpha}} (D_\mu A_i) \bar{\psi}^{\dot{\alpha}i} \right\}, \tag{49}$$

$$D = D^\mu \bar{A}^i D_\mu A_i + \frac{1}{2} i \bar{\psi}^{\dot{\alpha}i} (\sigma^\mu)_{\alpha\dot{\alpha}} \left( \bar{D}_\mu \psi_i^\alpha \right) - \bar{F}^i F_i. \tag{50}$$

Setting  $\psi_\alpha = F_i = 0$ , Eq. (39) is

$$\mathcal{L}_{susy} = -\frac{f_\pi^2}{8n^2} \bar{D}^\mu \bar{A} D_\mu A + \frac{1}{8e^2 n^2} \left[ -\frac{1}{2} F_{\mu\nu}^2 + (\bar{D}^\mu \bar{A} D_\mu A)^2 \right]. \tag{51}$$

We see that in Eq. (51) the second term is fourth-order in time derivatives. However, there are other possible fourth-order terms [5]

$$\square \bar{A} \square A - (\bar{D}^\mu \bar{A} D_\mu A)^2. \tag{52}$$

Thus, we can add Eq. (52) into Lagrangian density (51)

$$\mathcal{L}_{susy} = -\frac{f_\pi^2}{8n^2} \bar{D}^\mu \bar{A} D_\mu A + \frac{1}{8e^2 n^2} \left[ \alpha \left\{ -\frac{1}{2} F_{\mu\nu}^2 + (\bar{D}^\mu \bar{A} D_\mu A)^2 \right\} + \right.$$

$$+\beta \left\{ \square \bar{A} \square A - (\bar{D}^\mu \bar{A} D_\mu A)^2 \right\}. \quad (53)$$

The hedgehog ansatz is defined as [1, 2]

$$U^n(r) = \cos nf(r) + i\vec{\tau} \frac{\vec{r}}{r} \sin nf(r). \quad (54)$$

From Eq. (30), we have

$$A_1 = \cos nf(r) + i \cos \theta \sin nf(r), \quad (55)$$

$$A_2 = ie^{i\varphi} \sin \theta \sin nf(r). \quad (56)$$

Inserting Eqs. (55) and (56) into the supersymmetric Lagrangian density (53), we obtain the static energy

$$\begin{aligned} E = 4\pi \frac{f_\pi}{e} \int dx x^2 \left\{ \frac{1}{12} \left( f'^2 + \frac{2 \sin^2 nf}{n^2 x^2} \right) + \left( \frac{\alpha+\beta}{15} \right) \left( f'^2 - \frac{\sin^2 nf}{n^2 x^2} \right) \right. \\ \left. + \frac{\beta}{12} \left( f'' + \frac{2f'}{nx} - \frac{\sin 2nf}{n^2 x^2} \right) \right\}. \end{aligned} \quad (57)$$

From a minimum condition of the hedgehog's energy  $\delta_f E = 0$ , we obtain the following nonlinear differential equation of  $f(x)$

$$\begin{aligned} -x^2 f'' - 2x f' + \frac{\sin 2nf}{n} + \frac{4(\alpha+\beta)}{5} \left[ \frac{2f'' \sin^2 nf}{n^2} - 6x^2 f'^2 f'' - 4x f'^3 + \right. \\ \left. + \frac{f'^2 \sin 2nf}{n} + \frac{\sin^2 nf \sin 2nf}{n^3 x^2} \right] + \beta \left[ x^2 f^{(4)} + \frac{4x f^{(3)}}{n} - \frac{4f'' \cos 2nf}{n} + \right. \\ \left. + \frac{4f'^2 \sin 2nf}{n^2} - \frac{4 \sin^2 nf \sin 2nf}{n^3 x^2} \right] = 0, \end{aligned} \quad (58)$$

where  $x = ef_\pi r$  is the dimensionless variable.

Two boundary conditions of Eq. (58) are defined as

$$f(0) = \pi, \quad (59)$$

$$f(\infty) = 0. \quad (60)$$

Eq. (58) can not give the analytic solutions. So we use numerical method to solve it.

## 4 The gravitational composite Skyrme model

The Lagrangian composite Skyrme model coupled with gravity can be defined as

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{CS} \quad (61)$$

where  $\mathcal{L}_G = \frac{1}{16\pi G} R$ ,  $R$  is a curvature scalar and  $G$  is a Newton constant,

$$\mathcal{L}_{CS} = -\frac{F_\pi^2}{16n^2} g^{\mu\nu} \text{Tr} (\partial_\mu U^n \partial_\nu U^n) + \quad (62)$$

$$+ \frac{1}{32e^2n^4} g^{\mu\nu} g^{\rho\sigma} \text{Tr} [(\partial_\mu U^n) U^{-n}, (\partial_\rho U^n) U^{-n}] [(\partial_\nu U^n) U^{-n}, (\partial_\sigma U^n) U^{-n}], \quad (63)$$

$U(x)$  is an  $SU(2)$  chiral field,  $F_\pi$  is the pion decay constant,  $e$  is a dimensionless parameter. We consider the static spherically symmetric metric given by

$$ds^2 = -N^2(r) C(r) dt^2 + \frac{1}{C(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (64)$$

where

$$C(r) = 1 - \frac{2m(r)}{r}. \quad (65)$$

From the equation

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (66)$$

we can define tensor metric  $g_{\mu\nu}$

$$g_{\mu\nu} = \left( -N^2(r) C(r), \frac{1}{C(r)}, 1, 1 \right)_{diag} \quad (67)$$

$$\rightarrow \sqrt{-g} = N(r). \quad (68)$$

Inserting the hedgehog ansatz proposed by Skyrme [1]  $U(r) = \cos f(r) + i \vec{n} \vec{\tau} \sin f(r)$  into the Lagrangian density (1), we can obtain the static energy density for the chiral field

$$\mathcal{E}_S = \frac{F_\pi^2}{8} \left( C f'^2 + 2 \frac{\sin^2 n f}{n^2 r^2} \right) + \frac{1}{2e^2} \frac{\sin^2 n f}{n^2 r^2} \left( 2C f'^2 + \frac{\sin^2 n f}{n^2 r^2} \right), \quad (69)$$

where  $\vec{\tau}$  are Pauli matrices and  $\vec{n} = \frac{\vec{r}}{r}$ .

Introducing dimensionless variables, they are useful in solving a field equation

$$x = e F_\pi r, \quad (70)$$

$$\mu(x) = e F_\pi m(r). \quad (71)$$

In terms of  $x$  and  $\mu(x)$ , the static energy can be written as

$$E_S = \int \sqrt{-g} \mathcal{E}_S 4\pi r^2 dr, \quad (72)$$

or

$$E_S = 4\pi \frac{F_\pi}{e} \Gamma, \quad (73)$$

where

$$\Gamma = \int_0^\infty \left\{ \frac{1}{8} \left( e f'^2 + 2 \frac{\sin^2 n f}{n^2 x^2} \right) + \frac{\sin^2 n f}{2n^2 x^2} \left( 2C f'^2 + \frac{\sin^2 n f}{n^2 x^2} \right) \right\} N x^2 dx. \quad (74)$$

The energy tensor is defined as

$$T_{ij} = \frac{2}{\sqrt{-g}} \left[ \frac{\partial(\sqrt{-g} \mathcal{L}_G)}{\partial g^{ij}} - \frac{\partial}{\partial x^k} \frac{\partial(\sqrt{-g} \mathcal{L}_G)}{\partial g_k^{ij}} \right], \quad (75)$$

where

$$\frac{\partial \sqrt{-g}}{\partial g^{ij}} = -\frac{1}{2} \sqrt{-g} g_{ij}. \quad (76)$$



Inserting the form of Lagrangian density (61) into Eq. (76), we can get

$$T_{ij} = -\frac{1}{2n^2} F_\pi^2 \text{Tr} \left( L_i L_j - \frac{1}{2} g_{ij} L_\mu L^\mu \right) + \frac{1}{8n^2} \text{Tr} \left( F_{i\mu} F_{j\nu} g^{\mu\nu} - \frac{1}{4} g_{ij} F_{\mu\nu} F^{\mu\nu} \right), \quad (77)$$

where

$$L_\mu = U^{-n} \partial_\mu U^n, \quad (78)$$

$$F_{\mu\nu} = [L_\mu, L_\nu]. \quad (79)$$

The Einstein's gravitational field equation is

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij}, \quad (80)$$

or

$$R_{ij} g^{ij} - \frac{1}{2} g_{ij} R g^{ij} = 8\pi G T_{ij} g^{ij}. \quad (81)$$

For  $i = j = 0, 1$  we can get two equations for two unknown variables  $N(x)$ ,  $\mu(x)$

$$R_{00} g^{00} - \frac{1}{2} g_{00} R g^{00} = 8\pi G T_{00} g^{00} \quad (82)$$

$$\rightarrow N' = \frac{\alpha}{4} \left( x + \frac{8 \sin^2 n f}{n^2 x} \right) N f'^2, \quad (83)$$

$$R_{11} g^{11} - \frac{1}{2} g_{11} R g^{11} = 8\pi G T_{11} g^{11} \quad (84)$$

$$\rightarrow \mu' = \frac{\alpha}{8} \left[ \left( x^2 + 8 \frac{\sin^2 n f}{n^2} \right) C f'^2 + 2 \frac{\sin^2 n f}{n^2} + 4 \frac{\sin^4 n f}{x^2 n^4} \right]. \quad (85)$$

where  $\alpha = 4\pi G F_\pi^2$  is the coupling constant. The variation of the static energy of chiral field with respect to the profile  $f(x)$  leads to the field equation

$$\delta_f \mathcal{E}_S = 0, \quad (86)$$

$$\begin{aligned} & NC \left( x^2 + 8 \frac{\sin^2 n f}{n^2} \right) f'' + \left( x^2 + 8 \frac{\sin^2 n f}{n^2} \right) N' C f' - \left( 1 + 4 \frac{\sin^2 n f}{n^2 x^2} + 4 C f'^2 \right) N \sin 2n f + \\ & + 2 \left( x + 4 \sin 2n f f' \right) N C f'^2 + 2 \left( 1 + 8 \frac{\sin^2 n f}{n^2 x^2} \right) (\mu - \mu' x) N f' = 0. \end{aligned} \quad (87)$$

Boundary conditions of this equation are [1, 3]

$$f(0) = \pi, \quad (88)$$

$$f(\infty) = 0. \quad (89)$$

To describe nucleon states, we need to quantize classical skyrmion. The useful method that is used is a collective quantization. Consider collective coordinates are  $A(t) = a_0(t) + i\vec{\tau}\vec{a}(t)$  with conditions  $a_\mu a_\mu = 1$  and  $a_\mu \dot{a}_\mu = 0$  ( $\mu = 0, 1, 2, 3$ ). By this way, skyrmions to be real particles (as nucleon). With  $A(t)$ , the  $U^n(x, t)$  will transform as

$$U^n(x, t) = A(t) U_0^n(x) A^{-1}(t). \quad (90)$$

Putting Eq. (90) into Lagrangian of Einstein-Skyrme system (61), we can get

$$\mathcal{L} = -M + \lambda Tr \left( \dot{A} \dot{A}^{-1} \right) = -M + 2\lambda \sum_{\mu=0}^3 \dot{a}_{\mu}^2, \quad (91)$$

where  $M = E_S$  is classical skyrmion's mass (in the unit  $c = \hbar = 1$ ) and

$$\lambda = \frac{2\pi}{3F_{\pi}e^3} \Lambda, \quad (92)$$

$$\Lambda = \int_0^{\infty} \frac{1}{NC} \left[ 1 + 4 \left( Cf' + \frac{\sin^2 nf}{n^2 x^2} \right) \frac{x^2 \sin^2 nf}{n^2} \right] dx. \quad (93)$$

Following [3], the Hamiltonian is defined as

$$H = M + \frac{1}{8\lambda} \sum_{\mu=0}^3 \left( -\frac{\partial^2}{\partial a_{\mu}^2} \right). \quad (94)$$

Eigenvalues of the Hamiltonian (94) are

$$E = M + \frac{l(l+2)}{8\lambda}, \quad (95)$$

where  $l = 2I = 2J$  and  $(I, J)$  are respectively an isospin and a spin quantum number. For the nucleon ( $J = 1/2$ ) and the delta ( $J = 3/2$ ) we get

$$M_N = M + \frac{3}{8\lambda}, \quad (96)$$

$$M_{\Delta} = M + \frac{15}{8\lambda}. \quad (97)$$

From Eqs. (73), (74), (92), (93), (96), (97),  $e$  and  $F_{\pi}$  can be found as

$$F_{\pi} = \frac{1}{4\sqrt{6}} \sqrt[4]{\frac{\Lambda (M_{\Delta} - M_N) (5M_N - M_{\Delta})}{\pi^2 \Gamma^3}}, \quad (98)$$

$$e = \frac{16\pi F_{\pi} \Gamma}{5M_N - M_{\Delta}}. \quad (99)$$

The baryon current is defined as

$$B^{\mu} = \frac{\varepsilon^{\mu\nu\rho\sigma}}{24\pi^2 n} \frac{1}{\sqrt{-g}} Tr \left( U^{-n} \partial_{\nu} U^n U^{-n} \partial_{\rho} U^n U^{-n} \partial_{\sigma} U^n \right). \quad (100)$$

From Eq. (90), the baryon currents zeroth component (baryon number density) is found

$$B^0 = -\frac{1}{2\pi^2} \frac{1}{N} \frac{f' \sin^2 nf}{r^2}. \quad (101)$$

Using Eqs. (68), (88), (89) the baryon number becomes

$$B = \int_0^{\infty} 4\pi r^2 \sqrt{-g} B^0 dr = -\frac{2}{\pi} \int_{\pi}^0 \sin^2 nf df = 1. \quad (102)$$

The isoscalar mean square radius is defined in terms of the baryon number density

$$\langle r^2 \rangle_{I=0} = \int_0^\infty r^2 \sqrt{-g} B^0 4\pi r^2 dr = \int_0^\infty 4\pi r^4 N B^0 dr = -\frac{1}{(eF_\pi)^2} \frac{2}{\pi} \int_0^\infty x^2 f' \sin^2 nf dx. \quad (103)$$

The isoscalar magnetic square radius is defined as

$$\langle r^2 \rangle_{M,I=0} = \frac{3}{5} \frac{\int_0^\infty r^4 \sqrt{-g} B^0 4\pi r^2 dr}{\langle r^2 \rangle_{I=0}}. \quad (104)$$

The other components of baryons current are found

$$B^i = \frac{i\varepsilon^{ijk}}{2\pi^2} \frac{1}{\sqrt{-g}} \frac{\sin^2 nf}{r^2} f' x_k Tr \left[ \dot{A}^{-1} A \tau_j \right]. \quad (105)$$

From [3], we can get

$$B^i = \frac{\varepsilon^{ijk}}{2\pi^2} \frac{1}{\sqrt{-g}} \frac{\sin^2 nf}{r^2} f' x_k \frac{J_j}{2\Gamma}. \quad (106)$$

The time component of vector current  $V$  is defined as

$$V_i^0 = \frac{i}{3} \frac{\sin^2 nf}{n^2 r^2} \left[ F_\pi^2 + \frac{4}{e^2} \left( C f'^2 + \frac{\sin^2 nf}{n^2 r^2} \right) \right] Tr \left( \dot{A} A^{-1} \tau_i \right), \quad (107)$$

or

$$V_i^0 = \frac{\sin^2 nf}{3n^2 r^2} \left[ F_\pi^2 + \frac{4}{e^2} \left( C f'^2 + \frac{\sin^2 nf}{n^2 r^2} \right) \right] \frac{T_i}{2\Gamma}. \quad (108)$$

The density of charge of nucleon is defined by Gell-Mann and Nishijima

$$Q = \int_0^\infty \rho_{nu} dr = I_3 + \frac{1}{2} B, \quad (109)$$

where

$$B = \int_0^\infty 4\pi r^2 \sqrt{-g} B^0 dr = - \int_0^\infty \frac{2f' \sin^2 nf}{\pi} dr, \quad (110)$$

$$I_3 = \int_0^\infty \sqrt{-g} r^2 V_3^0 d\Omega dr = \int_0^\infty N \frac{4\pi \sin^2 nf}{3n^2} \left[ F_\pi^2 + \frac{4}{e^2} \left( C f'^2 + \frac{\sin^2 nf}{n^2 r^2} \right) \right] \frac{T_3}{2\Gamma} dr. \quad (111)$$

So

$$\rho_{nu} = N \frac{4\pi \sin^2 nf}{3n^2} \left[ F_\pi^2 + \frac{4}{e^2} \left( C f'^2 + \frac{\sin^2 nf}{n^2 r^2} \right) \right] \frac{T_3}{2\Gamma} - \frac{f' \sin^2 nf}{\pi}. \quad (112)$$

For  $T_3 = 1/2$  and  $T_3 = -1/2$  we get respectively the density of charge of proton and neutron. The isoscalar magnetic moment and isovector magnetic moment are defined as

$$\vec{\mu}_{I=0} = \frac{1}{2} \int \sqrt{-g} \left( \vec{r} \times \vec{B} \right) 4\pi r^2 dr, \quad (113)$$

$$\mu_{I=1} = \frac{1}{2} \int \sqrt{-g} \left( \vec{r} \times \vec{V}^3 \right) 4\pi r^2 dr. \quad (114)$$

According to [3], final results of Eqs. (113), (114) are defined

$$\mu_p = \frac{1}{4} \left[ \frac{4}{9} M_N (M_\Delta - M_N) \langle r^2 \rangle_{I=0} + \frac{2M_N}{M_\Delta - M_N} \right], \quad (115)$$

$$\mu_p = \frac{1}{4} \left[ \frac{4}{9} M_N (M_\Delta - M_N) \langle r^2 \rangle_{I=0} - \frac{2M_N}{M_\Delta - M_N} \right]. \quad (116)$$

## 5 Conclusions

We have constructed the term of mass of pion and calculated numerically in a case of  $n = 4$ . We have applied this model to define static properties of nucleon. A few obtained results are better than ones that predicted before [1, 3, 4, 9] as the axial coupling  $g_A = \mathbf{1.22}$  (in [1]:  $g_A = 1.3$ , in [3]:  $g_A = 0.61$ , in [4]:  $g_A = 0.65$ , in [9]:  $g_A = 1.27$ ), and  $\mu_{neu}(mag) = -\mathbf{1.57}$  (in [1]:  $\mu_{neu}(mag) = -1.17$ , in [3]:  $\mu_{neu}(mag) = -1.31$ , in [4]:  $\mu_{neu}(mag) = -1.24$ ). We have constructed successfully the formalism of supersymmetric composite Skyrme model. The proof was that the numerical solutions of field equation (24) were obtained. This was a role to calculate some static properties of nucleon. The formalism of gravitational composite Skyrme model was constructed. In fact, we have not calculated numerically the field equation and some properties of nucleon, but we believed that they will be performed in the near future. We also believed that our results could be applied in some fields of cosmology.

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\* figures for a case of pion's mass

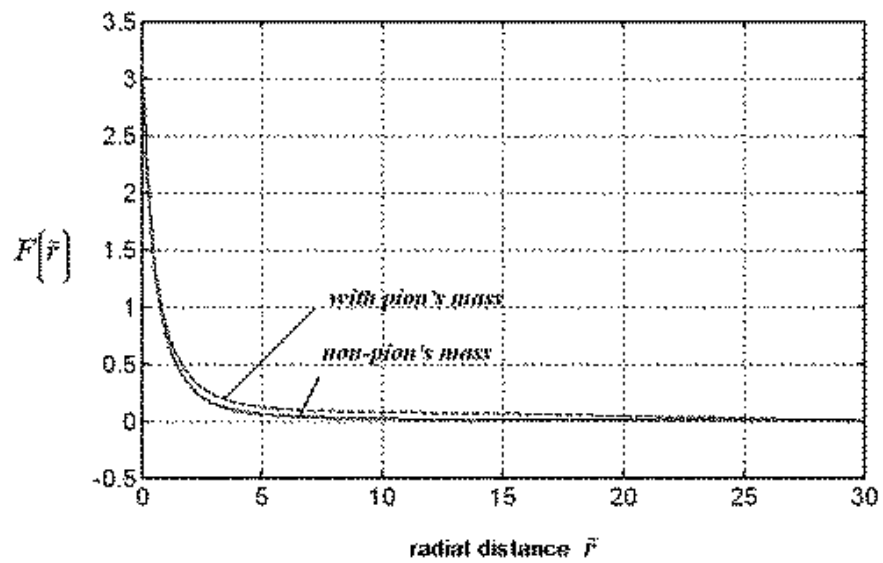


Figure 1:  $F(\tilde{r})$  as a function of the radial distance  $\tilde{r}$

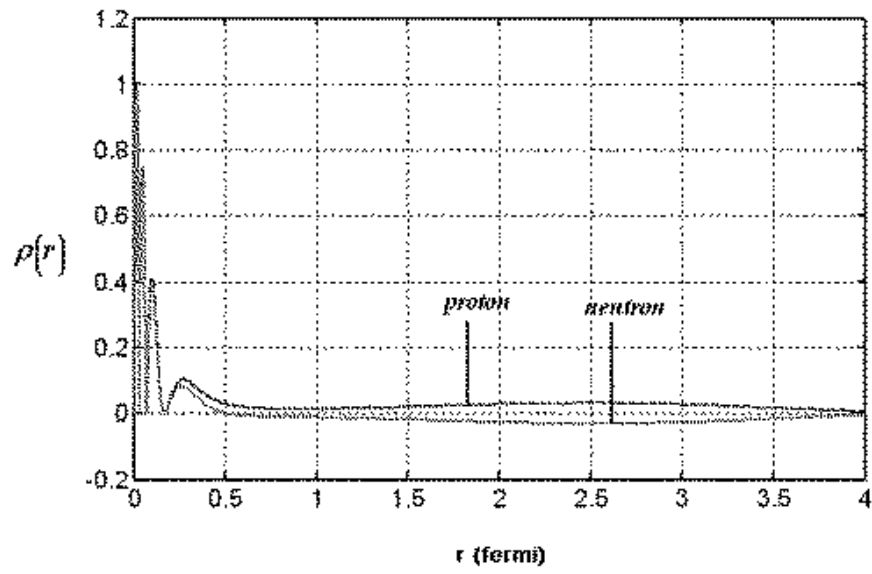


Figure 2: Neutron and proton charge densities  $\rho(r)$  as functions of the distance  $r(\text{fm})$

\*\* the figure for a case of supersymmetry

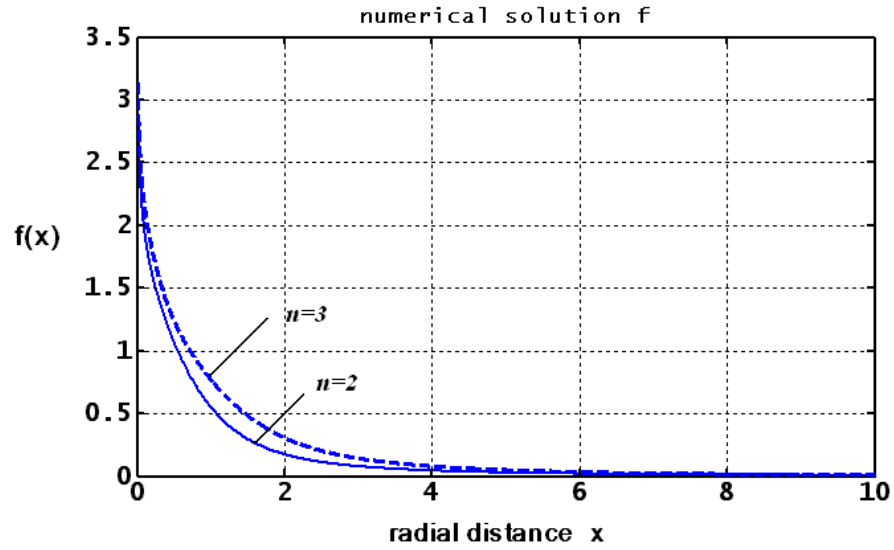


Figure 3: The numerical solution  $f(x)$  at  $\alpha = 1, \beta = 0, n = 2, 3$